Color Symmetry in Plane Patterns

Frank A. Farris
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The symmetries of a pattern form a group

A symmetry is a transformation that leaves the pattern invariant

Rosettes, friezes, wallpaper
My origin story: grating def’ns of “pattern”

“A frieze pattern is a set of points that is invariant under...”

Set of points? A pattern is a function!
Stumbled on color symmetry by accident

Rendered in Excel!
My origin story: grating def’ns of “pattern”

“A pattern is obtained by repeating a motif...”
My response: Patterns are made from waves!
SymmetryWorks software allows you to play too.
Public domain software written by students at Bowdoin College (and SCU)
Connect function-theoretic approach to color symmetry

\[ f(g(z)) = f(z) \text{ for } g \in G \]
Read the details:

CREATING SYMMETRY
The Artful Mathematics of Wallpaper Patterns
Frank A. Farris
• Classifying patterns
• Two-color symmetry
• Three-color symmetry

Connect to textiles in the real world?
Classifying patterns

**Wallpaper:**
Translational symmetry along two independent axes
Classify patterns by symmetries
Too many to list!

Mirrors

4-Centers

Glide reflections
Group concept: Compose two symmetries, get a symmetry

“freedom and constraint”
Learn how to draw a “fundamental cell”
Complete cell diagram with…
Group concept: a few symmetries can generate all

This group is called $p4g$ by the International Union of Crystallographers.
p4g can be generated by one mirror, one rotation
Two patterns have the same “type” if their symmetry groups are the same.
Wallpaper surprise:

Exactly 17 types

Cell diagrams for the 17 wallpaper groups
Chalk Slam at Carleton
See them everywhere!

pmg pattern from The Shining
Construct wallpaper functions

\[ f(g(z)) = f(z) \text{ for } g \in G \]
SymmetryWorks screenshot

Written by students at Bowdoin College, based on software by SCU students. Handout available.
A few favorites: Minimalist

Recognize p4g?
A few favorites: Fanciful

Machine learning project: Find patterns that humans will like
Color-reversing (or 2-color) symmetry

\[ f(g(z)) = f(\bar{z}) \text{ for some} \]
\[ f(h(z)) = -f(\bar{z}) \text{ for others} \]
Requires a color-reversing wheel

And the right waveforms
A break for algebra: Suppose

\[ f(h(z)) = -f(z) \text{ for some transformation } h \]

\[ G_c = \{ g \mid f(g(z)) = \pm f(z) \} \]

\[ G = \{ g \mid f(g(z)) = f(z) \} \]

Color gp
Sym. gp

Homomorphism

\[ \phi : G_c \to \{1, -1\} \]

\[ G \text{ is normal in } G_c \text{ of index 2} \]

\[ f(g(z)) = \phi(g)f(z) \text{ for all } g \in G_c \]

How many ways?
17 symmetry types
46 color-reversing types

Actual symmetry group: p3

Color-reversing or -preserving symmetries: p6

Type: p6/p3

Color-reversing half turn
Actual symmetry group: pmg
Color-reversing or -preserving symmetries: cmm
Type: cmm/pmg
Color black counts as 0
THE GEOMETRICAL BASIS OF PATTERN DESIGN. Part IV: Counterchange Symmetry in Plane Patterns, by H. J. Woods

Journal of Textiles (Manchester) 1936

Figures from Grünbaum and Shephard, *Tilings and Patterns*

Ad hoc notations like $p4g[2]_2$
Figures from appendix of *Creating Symmetry*

Types p4m/cmm and p4g/cmm
Recipes for 63 types

These recipes are not encoded in current software
Three-color (color-turning) symmetry

\[ f(g(z)) = \phi(g)f(z) \text{ for all } g \in G_c \]

\[ f(\omega z) = \omega f(z) \]
Three-color (color-turning) symmetry

Diagonal 1/3 translations are color-turning
Physical object not invariant under mirror symmetry, but rather under half-turn symmetry in space.
Connect to the non-digital world?

“Woven” polyhedral shapes
Aside: Polyhedral Sampler from 2015
Icos from five tetrahedra
60 bands.
Does not have 5-color symmetry
Question: Which frieze and wallpaper patterns can be woven in chain mail? Color patterns?